SEQUENCES AND PATTERNS

GMAT will test your ability to recognize patterns and iterations, and to determine the value for an element in a sequence. 'Recognition' of a **pattern** is a critical skill that you will require in real life too. If you consistently fail to win the lottery week after week, there is a pattern of picking a bad set of numbers. If a sales person consistently fails to meet his quota, there is a pattern of non-performance. If one's lover consistently fails to send flowers on Valentine's day, there is a pattern of 'couldn't-care-less' aspect to his or her behavior. Normally, we will 'recognize' such patterns and make a 'federal case' out of them. GMAT problems will ask you to do the same in the context of numbers and variables.

Let us say that we have a sequence of numbers as follows:

2, 3, 5, 7, 11, 13, 17, 23, X, ……..

We can see that the above is a sequence of positive prime integers. The next prime integer after 23 must be 29. Therefore, we can logically conclude that X must be 29.

Let us take a look at another sequence of values:

1, 2, 4, 8, 16, 32, X,…..

We can see that each successive value is double the preceding one. We must logically conclude that X must be twice 32 or 64.

Consider the following problem:

"In a recurring sequence, -4, -2, 0, 2, 4 are values that repeat indefinitely in the same order. What is the sum of the 27th value and the 29th value in the sequence?"

Because the 5 values specified repeat ad infinitum in the same order, we can see that the 26th value must start in –4. The 27th value will be –2. The 28th value will be 0. The 29th value is 2. **The sum of the 27th value and the 29th value is –2+2 = 0**. We must choose the option that corresponds to a value of 0 for the sum of the 27th and the 29th terms.

Consider another 'higher difficulty level problem':

"If in a sequence $T_1 = 2$, $T_2 = 5$, $T_3 = 11$, $T_4 = 23$, $T_5 = 47$, What is T_7? What is the expression for T_N in terms of the preceding values in the sequence?"

Do you get the feeling that you just walked out of a hot tub in a sauna room? You will not be alone and will have plenty of company in this regard. But you must bear in mind that no GMAT problem is meant to be difficult. You must also bear in mind that GMAT problems test your 'reasoning' ability and your 'ability to recognize patterns'.

What do we see is going on here?
We see that T_2 is one more than TWICE the preceding term T_1. T_3 is one more than twice the preceding term T_2. T_4 is one more than TWICE the preceding term T_3. T_5 is one more than TWICE the preceding term T_4 and so on. Can you see a pattern emerging here?
We can set up the Formula as follows:

$$T_N = 1 + 2\,(T_{N-1})$$

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The value for T_7 will be 1 more than TWICE T_6. The value for T_6 will be 1 more than TWICE T_5. Knowing that T_5 is 47, we can conclude that $T_6 = 1 + 2(T_5)$

$= 1 + 2(47) = 95$

Therefore, $T_7 = 1 + 2(T_6)$

$\mathbf{T_7 = 1 + 2\ (95) = 1 + 190 = 191}$

NOTE: We could have also set up the ***FORMULA*** representing the pattern as follows:

$T_1 = 2$

$T_2 = T_1 + 3\bullet 2^0$

$T_3 = T_2 + 3\bullet 2^1$

$T_4 = T_3 + 3\bullet 2^2$

$T_5 = T_4 + 3\bullet 2^3$

The pattern is as described in the rows above.

We can 'extrapolate' the above sequence and make the following 'logical determinations'.

Each successive term after the first term is the sum of the preceding term and a multiple of 3 value. Notice that T_5 is the sum of the preceding term T_4 and the exponent on the base of 2 is two less than the suffix of the term T_5. You can see that this pattern holds true for all the values after the first value.

We can, therefore, conclude that T_N must be equal to:

$$\mathbf{T_N = T_{N-1} + 3\bullet 2^{N-2}}$$

We can test this equation by replacing N with 2 and by checking to see whether we get a value of 5 for the Term 2. Similarly, replace N with 3 and check to see whether you get a value of 11 for the term 3. You will.

Now we can determine T_7 by simply replacing N with 7 in the above relationship.

$T_7 = T_6 + 3\bullet 2^5 = T_6 + 3\bullet 32 = T_6 + 96$

$T_6 = T_5 + 3\bullet 2^4 = 47 + 3\bullet 16 = 95$

Therefore, $\mathbf{T_7 = 95 + 96 = 191}$

Notice that the second approach is infinitely more abstruse and GMAT or GRE is likely to have you come up with the FORMULA as discussed in the first scenario. But, you will not get both the formulas because they are both correct and you cannot pick one over the other.

Can you see how this 'pattern' problems are dealt with? Let us do a couple of more so that you can get 'real comfortable' with this 'pattern stuff'. *(Never mind the fact that we used an adjective 'real' to modify another adjective, 'comfortable'. We are not dealing with Sentence Correction here, are we? If it were sentence correction, we would have used 'really comfortable')*

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PROBLEM 1

"If F(X) = 1/X – 1/(X+1),
What is the sum of F(1)+F(2)+F(3)+F(4)+…….+F(100)?"

The problem looks 'intimidating' but we must bear in mind that no problem on the test is meant to be difficult. Challenging, perhaps, but never 'difficult'. We notice that the original statement is in terms of X, but the summation involves real integers.

Let us try to replace the variable X with the integers and see whether there is a recognizable pattern emerging.

F(1) = 1/1 – ½ = ½
F(2) = ½ - 1/3 = 1/6
F(3) = 1/3 – ¼ = 1/12
F(4) = ¼ - 1/5 = 1/20

We notice that the problem will be 'very complex' if it involves adding fractions such as ½, 1/6, 1/12, 1/20, and so on. GMAT is not a slave-driver but a test that asks you to work through the problems efficiently and smartly. We must say to ourselves that 'adding fractions' is not the way to go. Let us 'size up' the problem and see whether there is a pattern emerging. We do see a pattern. We notice that the –1/2 is the mirror image of ½; -1/3 is the mirror image of 1/3 and so on. When we add F(1), F(2), etc., each successive term will cancel out leaving just the first term of the first function and the last term of the last function. Notice that the successive terms cancel out diagonally.

F(1) = 1 – ½
+
F(2) = ½ -1/3
+
F(3) = 1/3 – ¼
+
F(4) = ¼ - 1/5
+
F(5) = 1/5 – 1/6
+
F(6) = 1/6 – 1/7
+
*
*
*
*
+
F(99) = 1/99 – 1/100
+
F(100) = 1/100 – 1/101

The required sum = 1/1 – 1/101 = 101/101 – 1/101 = 100/101

Our ability to recognize an emerging 'pattern' saved the day for us, and got us to the answer. Train yourself to recognize patterns. The test will test your ability to do so.

> Also, the above problem tested your ability to 'associate' values with X. Given that F(X) is in terms of X, we 'associated' X with 1 when we computed F(1); X with 2 when we computed F(2), and so on. If we are given that $F(X,Y) = X^3 + Y^2$, we can compute F(-3, -2) by 'associating' X with –3 and Y with –2.
> $F(-3, -2) = (-3)^3 + (-2)^2 = -27 + (-4) = -31$

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Let us take a look at another problem asking us to recognize patterns.

"In a sequence, the first three values are the first three positive prime integers in ascending order, and each of the successive values is the sum of all of the preceding values. What is the ratio of the 30th term to the 25th term?"

We have the sequence going as follows:

2, 3, 5, (2+3+5), (2+3+5+2+3+5), ……

Notice that a pattern begins to emerge. The fifth term is twice the fourth, and we will expect the same pattern to continue. Let us verify that it is indeed so.

The sequence is:

2, 3, 5, 10, 20, (20+10+5+3+2) 40, (40+20+10+5+3+2) 80, ……..

Each successive term after the third term is twice the preceding term, and the same iteration will continue.

Therefore, the term 26th will be twice term 25th and so on.

$Term_{26} = 2 \bullet T_{25}$
$T_{27} = 2 \bullet T_{26} = 2 \bullet 2 \bullet T_{25}$
$T_{28} = 2 \bullet T_{27} = 2 \bullet 2 \bullet 2 \bullet T_{25}$
$T_{29} = 2 \bullet T_{28} = 2 \bullet 2 \bullet 2 \bullet 2 \bullet T_{25}$
$T_{30} = 2 \bullet T_{29} = 2 \bullet 2 \bullet 2 \bullet 2 \bullet 2 \bullet T_{25} = 2^5 \bullet T_{25}$

Therefore, the ratio of T_{30} to T_{25} is $2^5 = 32/1$.

Take a look at another problem:

"If $S_1 = 9$, $S_2 = 99$, $S_3 = 999$, $S_4 = 9999$, …….What is S_N in terms of N?"

What is the pattern we recognize here? We notice that all the values are in terms of digit '9', which repeats as many times as the value of the suffix. S_1 is a single digit number 9. S_2 is a two digit number 99 and so on. We can determine that the digit 9 must recur N times in the value for S_N. We also see another pattern:
9 is 10-1
99 is $10^2 - 1$
999 is $10^3 - 1$
9999 is $10^4 - 1$

We can clearly see that S_N must be
$\mathbf{10^N - 1}$.

If you are unable to leap forward to this conclusion, you can 'back-solve' by looking at the options and by replacing N with 1 and checking to see whether you get a value of 9; Verify by replacing N with 2 and see that you get 99, and so on.

Recognition of 'patterns' can also be tested in data sufficiency problems. Take a look at the above problem in the following data sufficiency setting.

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DATA SUFFICIENCY

"If the set S contains subsets S_1 through S_N such that $S_1 = 9$, $S_2 = 99$, $S_3 = 999$, …. $S_N = 10^N - 1$, are all the subset values divisible by the prime integer P?"

1. $P < 5$
2. Subset S_5 is divisible by P.

The information provided is mind-numbing for sure, but we must repeat the mantra that GMAT problems are not supposed to be 'difficult' but 'challenging'. Also, keep in mind the simple rule that the harder the problem sounds, the less complex it will be.

How do we know that this problem is not as complex as it seems on first blush? Our reasoning says so. Remember that all problems will involve some degree of reasoning effort on your part. We notice a pattern in the subsets: They are all in terms of the recurring digit 9. Given that P is a prime integer, the only prime integer that will divide ALL of the subsets is 3. No other prime integer will evenly divide the values in the subsets. In fact, if the single-digit sum of the digits in an integer is 9, then the integer must be divisible by at least 1, 3, and 9. Knowing that 1 and 9 are not prime integers, we need to seek information telling us whether P is equal to 3 or not.

Notice how we 'transformed' a complex problem to a 'manageable' one through a simple reasoning exercise. Let us examine the statements one by one.

Statement 1 tells us that $P < 5$. Given that P is a prime integer, P could be 2 or 3. If P is 2, then none of the values of the subsets is divisible by P. On the other hand, if P is 3, then all the subset values are divisible by P. Because we do not know whether P is 2 or 3, we are unable to make a unique determination of whether all the subset values are divisible by P or not. We have conflicting outcomes across two different scenarios when we test $P = 2$ and $P = 3$. We must conclude that Statement 1 alone is not sufficient, and eliminate options A and D. These two choices cannot be picked if the statement 1 turns out to be not sufficient.

Statement 2 is somewhat intimidating but we must once again tell ourselves that the more intimidating the statement looks, the less complex it is likely to be.

We know that $S_5 = 99999$. How do we know this? The pattern tells us so. If S_5 is divisible by P, then P could be 3 or 41, (not to mention any other prime number that could be a factor of 99999) because both 3 and 41 are factors of 99999. Statement 2 is not sufficient because if P=3, then all the sets are divisible by P but if P=41, then all the sets are not divisible by P. We must, therefore, conclude that Statement 2 alone is not sufficient.
We now have to combine the two statements and determine whether all the sets are divisible by P. Knowing that $P < 5$ from Statement 1 and knowing that P could be at least 3 or 41 from Statement 2, we can conclude that P must be 3, the only value that is consistent with both statements 1 and 2. **We must, therefore, choose option C.**

The upshot of the above analysis is the following: "Do not get intimidated by how the problem looks and sounds. Just think through it, and remember: the more difficult and intimidating the problem seems the less complex it is likely to be."

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Let us plow through another Patterns problem and increase our comfort level with such problems.

The infinite sequence $a_1, a_2, \ldots, a_n, \ldots$ is such that $a_1 = 2$, $a_2 = -3$, $a_3 = 5$, $a_4 = -1$, and $a_n = a_{n-4}$ for $n > 4$. What is the sum of the first 97 terms of the sequence?

A. 72
B. 74
C. 75
D. 78
E. 80

In order to be able to see a pattern, we need to take some 'baby-steps' and set up a few more values of the 'sequence'.

$a_5 = a_{5-4} = a_1 = 2$
$a_6 = a_{6-4} = a_2 = -3$
$a_7 = a_{7-4} = a_3 = 5$
$a_8 = a_{8-4} = a_4 = -1$
$a_9 = a_{9-4} = a_5 = a_1 = 2$

As you can see, the sequence reverts to the first value '2' after a set of 4 values, and the sequence carries on in this fashion. We have seen a clear pattern of 'ad infinitum' repetition of the first 4 values. We can logically conclude that the 97th value will revert to 2. In the first 96 values, there are 24 sets of the first four values. The first 4 values add up to 3. The sum of the first 96 values is 24 times 3 = 72. If we add the 97th value to this total of 72, we get 74. Therefore, the sum of the first 97 values is 74. We must pick option B.

Let us take a look at another 'pattern' problem.

A certain computer program generates a sequence of numbers $a_1, a_2, \ldots$, an such that $a_1 = a_2 = 1$ and $a_k = a_{k-1} + 2a_{k-2}$ for all integers k such that $3 \le k \le n$. If $n > 6$, then $a_7 = ?$

A. 32
B. 43
C. 64
D. 100
E. 128

This is a relatively simpler problem than the ones we have been previously grappling with.

We can see that if k = 3, then
$a_3 = a_2 + 2a_1 = 1 + 2(1) = 3$
$a_4 = a_3 + 2a_2 = 3 + 2(1) = 5$
$a_5 = a_4 + 2a_3 = 5 + 2(3) = 11$
$a_6 = a5 + 2a_4 = 11 + 2(5) = 21$
$a_7 = a_6 + 2a_5 = 21 + 2(11) = 43.$
Option B is the answer.

Take another pattern problem for a test-drive. (Go to next page)

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If the sequence $x_1, x_2, x_3, \ldots, x_n, \ldots$ is such that $x_1 = 3$ and $x_{n+1} = 2x_n - 1$ for $n \geq 1$, then $x_{20} - x_{19} =$

A. 2^{19}
B. 2^{20}
C. 2^{21}
D. $2^{20} - 1$
E. $2^{21} - 1$

Let us take some 'baby-steps' and set up a few values of the sequence so that we can begin to see a clear 'pattern'.

$X_1 = 3$
$X_2 = 2x_1 - 1 = 2(3) - 1 = 5$
$X_3 = 2x_2 - 1 = 2(5) - 1 = 9$
$X_4 = 2x_3 - 1 = 2(9) - 1 = 17$
$X_5 = 2x_4 - 1 = 2(17) - 1 = 33$

Notice that the question is about the 'difference' of a particular set of consecutive values.

On the basis of the above representation of sequence, let us see whether any 'clear' pattern emerges about the 'difference' of consecutive values.

$x_2 - x_1 = 5 - 3 = 2^1$
$x_3 - x_2 = 9 - 5 = 2^2$
$x_4 - x_3 = 17 - 9 = 2^3$
$x_5 - x_4 = 33 - 17 = 2^4$

We are now able to see a clear pattern: the difference of the first two values is 2, that of next two is 2^2, that of the next two is 2^3, and so on. We also see that the exponent corresponds to the subscript of the value being subtracted. We can logically conclude that $x_{20} - x_{19}$ must be equal to 2^{19}, with the exponent on 2 coinciding with the subscript of the value x_{19} being subtracted. We must choose option A.

Pattern problems test your ability to **observe** and to **see a clear and recurring connection among the values . If, in the above problem, the question was: what is x_{20}**?, then we will use the sequence 3, 5, 9, 17, 33, …. to see the following pattern: (2^1+1), $(2^2 + 1)$, $(2^3+)1$, $(2^4+)1$, $(2^5+)1$, and so on. We can see that the exponent 1 corresponds to value 1, exponent 2 to value 2, and so on. We can logically conclude that x_{20} must be **$(2^{20}+ 1)$.**

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